Infinite Quasi-Exactly Solvable Models

H.D. Doebner[1]

Arnold Sommerfeld Institute for Mathematical Physics, Thechnical University of Clausthal,
Leibnizstrasse 10, 38-678 Clausthal, Germany

K. Łazarow

Department of Theoretical Physics, University of Łódź,
Pomorska 149/153, 90-236 Łódź, Poland

and

A.G. Ushveridze[2]

Department of Theoretical Physics, University of Łódź,
Pomorska 149/153, 90-236 Łódź, Poland
and
Institute of Theoretical Physics, Technical University of Clausthal,
Arnold Sommerfeld Str. 6, 38-678 Clausthal, Germany

Abstract

We introduce a new concept of infinite quasi-exactly solvable models which are constructable through multi-parameter deformations of known exactly solvable ones. The spectral problem for these models admits exact solutions for *infinitely many* eigenstates but not for the whole spectrum. The hermiticity of their hamiltonians is guaranteed by construction. The proposed models have quasi-exactly solvable classical conterparts.

---

[1]E-mail address: asi@pt.tu-clausthal.de
[2]E-mail address: alexush@mvii.uni.lodz.pl and alexush@krysia.uni.lodz.pl

1 Introduction

In this paper we present a simple method for constructing multi-dimensional Schrödinger equations admitting exact solutions for *infinitely many* eigenvalues and corresponding eigenfunctions. However, these equations cannot be solved exactly for the whole spectrum. This fact suggest to consider them as generalizations of known quasi-exactly solvable problems discovered several years ago [1, 2, 3, 4].

Remember that the distinguishing property of usual quasi-exactly solvable problems is that the number of their explicit solutions is finite [5-12]. For this reason the phenomenon of quasi-exact solvability has a purely quantum nature and does not admit any reasonable extension to the classical case.

A remarkable feature of models which we intend to discuss in this paper is that they have direct *classical counterparts* and the method for their construction we propose works equally well in both the classical and quantum cases.

In order to formulate the idea of this method it is reasonable to start with the simpler classical case.

Let H_0 be a Hamilton function of an integrable classical mechanical system whose solutions are explicitly known. Consider a deformation of H_0 given by the formula

$$H = H_0 + \sum_{i,k=1}^{d} (L_i - \lambda_i) U_{ik} (L_k - \lambda_k). \tag{1.1}$$

Here L_i are some independent functions on the phase space being in involution with each other and with the undeformed Hamilton function H_0,

$$\{H_0, L_i\} = 0, \qquad \{L_i, L_k\} = 0, \tag{1.2}$$

λ_i are arbitrary real parameters and U_{ik} is a matrix of arbitrarily chosen functions which are not assumed to be in involution with H_0.

The classical equations of motion for the deformed Hamilton function H can be explicitly solved at least for a part of trajectories. The proof is simple. Consider a level surface S determined by the system of equations $L_i = \lambda_i$, $i = 1, \ldots, d$. Because of the relations (1.2), S is an invariant surface for the undeformed system with Hamilton function H_0. Now consider the equation of motion for the deformed system with Hamilton function H. It has the form $\dot{f} = \{H, f\}$ or, in more detail,

$$\dot{f} = \{H_0, f\} + \sum_{i,k=1}^{d} (L_i - \lambda_i)(L_k - \lambda_k)\{U_{ik}, f\} + 2 \sum_{i,k=1}^{d} (L_i - \lambda_i) U_{ik} \{L_k, f\}. \tag{1.3}$$

The second and third terms in the right hand side of (1.3) vanish on the surfase S. This means that equations of motion and hence their *solutions* for deformed and undeformed systems coincide in the case of equal initial conditions on S. Because the undeformed system has been assumed to be integrable, the integrability of the deformed system on the level surface S is proved. But the trajectories lying on S form only a part of all possible trajectories of the deformed system. This means that the system with Hamilton function H is *partially integrable*.

The extensibility of these reasonings to the quantum mechanical case is obvious. Indeed, let $\hat{H}_0$ be now the hamiltonian of an algebraically solvable quantum mechanical system. Consider a deformation of the hamiltonian $\hat{H}_0$ given by the formula

$$\hat{H} = \hat{H}_0 + \sum_{i,k=1}^{d} (\hat{L}_i - \lambda_i) \hat{U}_{ik} (\hat{L}_k - \lambda_k). \tag{1.4}$$

Now $\hat{L}_i$ denote some hermitian operators[3] in Hilbert space commuting with each other and with the undeformed hamiltonian $\hat{H}_0$,

$$[\hat{H}_0, \hat{L}_i] = 0, \qquad [\hat{L}_i, \hat{L}_k] = 0, \tag{1.5}$$

---

[3]Hereafter the hermiticity will mean the essential self-adjoitness.

λ_i are arbitrary real parameters and $\hat{U}_{ik}$ is a symmetric matrix of arbitrarily chosen hermitean operators which do not generally commute with $\hat{H}_0$. The operator $\hat{H}$ constructed in such a way is again a hermitean operator and thus can be viewed as a hamiltonian for a cerain quantum mechanical model.

Let us now prove that the quantum spectral equations for this model can be explicitly solved at least for a part of the spectrum. The proof is essentially the same as in the classical case. Because of the relations (1.5), the operators $\hat{H}_0$ and $\hat{L}_i$ admit common invariant eigensubspaces. Denote by $\hat{S}$ an invariant subspace corresponding to fixed eigenvalues m_i of operators $\hat{L}_i$. It is absolutely obvious that if the parameters λ_i coincide with these eigenvalues m_i, then the extra (deformation) term in the operator $\hat{H}$ vanishes on all functions belonging to $\hat{S}$. This means that the spectral problem for $\hat{H}$ in $\hat{S}$ reduces to the spectral problem for $\hat{H}_0$ which, by assumption, is exactly solvable. This completes the proof.

The method described above gives an infinite series of quasi-exactly solvable problems. Indeed, the phenomenon of partial solvability appears always when the parameters λ_i coincide with the eigenvalues m_i of operators $\hat{L}_i$. Another feature of these models is that for any fixed m_i they form functionally large classes which follows from the arbitrariness of the entries of matrix $\hat{U}_{ik}$

If we want to realize this scheme restricting ourselves to second-order differential operators $\hat{H}_0$ and $\hat{H}$, then $\hat{L}_i$ should be first order operators and $\hat{U}_{ik}$ — ordinary functions.

In next sections of the paper we demonstrate that such models do really exist and lead to a large variety of multi-dimensional partially solvable quantum models defined on compact curved manifolds. The remarkable feature of these models is that they can be considered as models of charged particles situated in an external electrostatic field.

2   Some useful formulas

We start our discussion with the investigation of transformation properties of hermitean multi-dimensional second-order differential operators on compact manifolds.

Let $\mathcal{M}$ be some smooth compact manifold without boundaries described by coordinates $\xi = (\xi_1, \ldots, \xi_n)$ and characterized by the covariant metric tensor $\mathcal{G}^{\mu\nu}(\xi)$. Denote by $\mathcal{W}$ the Hilbert space of functions on $\mathcal{M}$ whose scalar product is given by the formula

$$\langle \Psi_1, \Psi_2 \rangle = \int_{\mathcal{M}} \Psi_1^*(\xi) \Psi_2(\xi) \frac{d\xi}{\sqrt{\mathcal{G}(\xi)}} \tag{2.1}$$

with $\mathcal{G}(\xi) \equiv \det \|\mathcal{G}^{\mu\nu}(\xi)\|$. Consider a second order differential operator

$$\mathcal{H} = -G^{\mu\nu}(\xi) \frac{\partial^2}{\partial \xi^\mu \partial \xi^\nu} - F^\mu(\xi) \frac{\partial}{\partial \xi^\mu} - E(\xi) \tag{2.2}$$

and assume that it is hermitian in $\mathcal{W}$. Then necessarily:

1. Functions $G^{\mu\nu}(\xi)$ are real while functions $F^\mu(\xi)$ and $E(\xi)$ may be complex.

2. The following relations hold:

$$\frac{\partial}{\partial \xi^\mu} G^{\mu\nu}(\xi) - G^{\mu\nu}(\xi) \frac{\partial}{\partial \xi^\mu} \ln \sqrt{\mathcal{G}(\xi)} = \text{Re } F^\nu(\xi), \tag{2.3}$$

$$\frac{\partial}{\partial \xi^\mu} \text{Im } F^\mu(\xi) - \text{Im } F^\mu(\xi) \frac{\partial}{\partial \xi^\mu} \ln \sqrt{\mathcal{G}(\xi)} = 2\text{Im } E(\xi). \tag{2.4}$$

If functions $F^\mu(\xi)$ and $E(\xi)$ are real then the second relation disappears.

3. If the matrix $G^{\mu\nu}(\xi)$ is positive definite so that $G(\xi) = \det\|G^{\mu\nu}(\xi)\|$ is a positive function on $\mathcal{M}$ then there exists a homogeneous transformation

$$H = \Phi^{-1}\mathcal{H}\Phi \tag{2.5}$$

realized by real function

$$\Phi = \left(\frac{\mathcal{G}(\xi)}{G(\xi)}\right)^{1/4} \tag{2.6}$$

which transforms the operator $\mathcal{H}$ into the form

$$H = -\sqrt{G(\xi)}\left(\frac{\partial}{\partial\xi^\mu} - \imath A_\mu(\xi)\right)\left\{\frac{G^{\mu\nu}(\xi)}{\sqrt{G(\xi)}}\left(\frac{\partial}{\partial\xi^\nu} - \imath A_\nu(\xi)\right)\right\} + V(\xi). \tag{2.7}$$

Here

$$A_\mu(\xi) = \frac{1}{2}G_{\mu\nu}(\xi)\mathrm{Im}\ F^\nu(\xi) \tag{2.8}$$

and

$$
\begin{aligned}
V(\xi) \quad &= \quad -G^{\mu\nu}(\xi)\left[A_\mu(\xi)A_\nu(\xi) + \Phi^{-1}(\xi)\frac{\partial^2}{\partial\xi^\mu\partial\xi^\nu}\Phi(\xi)\right] \\
&\quad -\mathrm{Re}\ F^\mu(\xi)\Phi^{-1}(\xi)\frac{\partial}{\partial\xi^\mu}\Phi(\xi) - \mathrm{Re}\ E(\xi)
\end{aligned}
\tag{2.9}
$$

The operator H can be interpreted as hamiltonian of a charged quantum particle moving on a certain curved manifold M (which is described by the covariant metric tensor $G^{\mu\nu}(\xi)$), interacting with electrostatic field $A_\mu(\xi)$ and with external potential $V(\xi)$.

3   The undeformed algebraically solvable models

Let Γ be a compact finite-dimensional Lie group and Λ be the corresponding Lie algebra. The Weyl generators of algebra Λ associated with its simple roots π_i and positive roots α we denote by L_i^0 and $L_\alpha^\pm$, respectively. We have:

$$[L_i^0, L_k^0] = 0, \qquad [L_i^0, L_\alpha^\pm] = \pm(\pi_i, \alpha)L_\alpha^\pm \tag{3.1}$$

It is known that these generators can be realized as vector fields on the homogeneous space

$$\mathcal{M} = \Gamma/\Gamma_0, \tag{3.2}$$

where Γ_0 is a certain stationary subgroup of Γ. Introducing the coordinates ξ^μ on $\mathcal{M}$, we can write

$$L_i^0 = T_a^{0\mu}(\xi)\frac{\partial}{\partial\xi^\mu}, \qquad L_\alpha^\pm = T_a^{\pm\mu}(\xi)\frac{\partial}{\partial\xi^\mu} \tag{3.3}$$

Now note that the compactness of Γ implies the compactness of $\mathcal{M}$. This enables one to introduce on $\mathcal{M}$ a Γ-invariant metric $\mathcal{G}^{\mu\nu}(\xi)$ with elements given by the following simple formula:

$$\mathcal{G}^{\mu\nu}(\xi) = K^{ik}T_i^{0\mu}(\xi)T_k^{0\nu}(\xi) + \sum_\alpha\left[T_\alpha^{+\mu}(\xi)T_\alpha^{-\nu}(\xi) + T_\alpha^{-\mu}(\xi)T_\alpha^{+\nu}(\xi)\right]. \tag{3.4}$$

4

in which K^{ik} is the Cartan part of the Killing tensor. It is known that the neutral generators L_i^0 are anti-hermitean with respect to the scalar product (2.1) with metric (3.4),

$$(L_i^0)^+ = -L_i^0, \tag{3.5}$$

while the raising and lowering generators $L_\alpha^\pm$ are anti-hermitian conjugated

$$(L_\alpha^\pm)^+ = -L_\alpha^\mp. \tag{3.6}$$

Consider the operator

$$\mathcal{H}_0 = \sum h^{ik} L_i^0 L_k^0 + \sum f_\alpha^+ L_\alpha^+ L_\alpha^- + \sum f_\alpha^- L_\alpha^- L_\alpha^+ \tag{3.7}$$

in which h^{ik} and $f_\alpha^\pm$ are arbitrary real constants. Obviously, this operator is hermitean (by construction) with respect to the metric (2.1) and (3.4),

$$\mathcal{H}_0^+ = \mathcal{H}_0, \tag{3.8}$$

moreover, it is a second order differential operator of the form (2.2), and therefore, according to the theorem of the previous section, it always can be reduced to the form (2.7).

Let us now show that the obtained quantum model is exactly solvable. This means that the whole spectrum of its hamiltonian can be obtained algebraically. In order to demonstrate this fact, let us repeat the reasonings of ref. [7] and consider the quadratic Casimir operator for the group Γ:

$$\Delta = K^{ik} L_i^0 L_k^0 + \sum_\alpha \left[L_\alpha^+ L_\alpha^- + L_\alpha^- L_\alpha^+ \right]. \tag{3.9}$$

This operator, which can be interpreted as the Laplace operator on $\mathcal{M}$, is hermitian, non-degenerate and commutes with the hamiltonian $\mathcal{H}$. From this it follows that the Hilbert space in which the hamiltonian H acts splits into a sum of eigensubspaces of the operator Δ. Since Δ commutes with generators L_i^0 and $L_\alpha^\pm$, every such subspace forms a finite-dimensional representation of the group Γ. The basis functions in these representation spaces are the so-called "generalized spherical harmonics" whose concrete form can be constructed in local coordinates ξ are fixed. In the basis of the generalized spherical harmonics the hamiltonian $\mathcal{H}$ takes the block diagonal form. Each block has a finite dimension and is completely disconnected from all others. Therefore, the spectral problem for this hamiltonian breaks up into an infinite number of finite-dimensional spectral problems, each of which can be solved algebraically.

4 The deformed partially solvable model

Let us now construct a new (deformed) operator $\mathcal{H}$ given by the formula

$$\mathcal{H} = \mathcal{H}_0 + \sum_{i,k} (L_i^0 - \imath \lambda_i) U_{ik} (L_k^0 - \imath \lambda_k) \tag{4.1}$$

in which U_{ik} is an arbitrary real symmetric matrix whose entries are arbitrarily chosen smooth functions on the manifold $\mathcal{M}$ and λ_i are some real parameters. Note that the deformed operator $\bar{H}$ is again hermitean

$$\mathcal{H}^+ = \mathcal{H}, \tag{4.2}$$

it is again a second order operator and thus can be reduced to the form (2.7).

Now, however, we cannot claim any longer that this operator admits a complete algebraization of the spectral problem as it was in the case of $\mathcal{H}_0$. Indeed, the reasonings given in the previous section

for $\mathcal{H}_0$ do not work because the operator $\mathcal{H}$ does not commute with the Laplace operator Δ (i.e. with the second-order Casimir invariant of the group Γ). The reason for this is the presence of the extra deformation term containing the arbitrarily chosen functions U_{ik}.

Nevertheless, from the fact of commutativity of $\mathcal{H}_0$ with generators L_i^0 and the reasonings given in section 1 it follows that the operator $\mathcal{H}$ for some quantized values of parameters $\lambda_i = m_i$ (i.e coinciding with eigenvalues of operators L_i^0) admits a partial algebraization of the spectral problem in the sense that one can construct purely algebraically a certain set of its eigenvalues and eigenfunctions. The number of these solutions is obviously equal to the dimension of the corresponding common eigensubspace of operators L_i^0. But this dimension is equal to infinity because the number of eigenstates of operators L_i^0 having one and the same projections m_i of the "generalized angular momenta" is infinite.

This reasoning completes the exposition of our method for building partially solvable models in an external electrostatic field.

5   An example

Following general prescriptions of section 3, let us consider the case of algebra $so(3)$ with three generators L^0, L^+, L^- obeying the commutation relations

$$[L^+, L^-] = 2L^0, \qquad [L^0, L^\pm] = \pm L^\pm \tag{5.1}$$

Choosing the operator (3.7) in the form

$$\mathcal{H}_0 = a(L^0)^2 + b(L^+L^- + L^-L^+) + cL^0 \tag{5.2}$$

and rewriting the generators in the spherical coordinates as

$$L^+ = (\cos\phi + i\sin\phi)\frac{\partial}{\partial\theta} + \mathrm{ctg}\theta(i\cos\phi - \sin\phi)\frac{\partial}{\partial\phi} \tag{5.3}$$

$$L^- = (-\cos\phi + i\sin\phi)\frac{\partial}{\partial\theta} + \mathrm{ctg}\theta(i\cos\phi + \sin\phi)\frac{\partial}{\partial\phi} \tag{5.4}$$

$$L^0 = -i\frac{\partial}{\partial\phi} \tag{5.5}$$

we obtain for $\mathcal{H}_0$:

$$\mathcal{H}_0 = -2b\frac{\partial^2}{\partial\theta^2} - (a + 2\mathrm{ctg}^2\theta)\frac{\partial^2}{\partial\phi^2} - 2b\mathrm{ctg}\theta\frac{\partial}{\partial\theta} - ic\frac{\partial}{\partial\phi} \tag{5.6}$$

This model is algebraically solvable because it commutes with Laplasian (see general discussion in section 3). Let us now construct the deformed operator $\mathcal{H}$ given by formula (4.1). We know that, in this case, we can not expect any longer a complete algebraization of the spectral problem, but if λ coincides with eigenvalues m of the operator L^0, a partial algebraization will be realized. Let $\lambda = m$ where m satisfies the equation $L^0\Psi = m\Psi$. In this case $\mathcal{H}$ takes the form:

$$\mathcal{H} = (-a - 2\mathrm{ctg}^2\theta + U(\theta,\phi))\frac{\partial^2}{\partial\phi^2} - 2b\frac{\partial^2}{\partial\theta^2} - 2b\mathrm{ctg}\theta\frac{\partial}{\partial\theta}$$

$$-(-ic + \frac{\partial U(\theta,\phi)}{\partial\phi} - 2imU(\theta,\phi))\frac{\partial}{\partial\phi} - (-im\frac{\partial U(\theta,\phi)}{\partial\phi} - m^2U(\theta,\phi)) \tag{5.7}$$

where $U(\theta,\phi)$ is an arbitrarily fixed real function.

Operator $\mathcal{H}$ is hermitian in a space of suitable functions on the sphere and is endowed with a scalar product given by

$$\langle \Psi_1, \Psi_2 \rangle = \int_{S^2} \Psi_1^*(\theta, \phi) \Psi_2(\theta, \phi) \sin \theta d\theta d\phi. \tag{5.8}$$

According to the results of section 2, (5.7) can be represented in the form (2.7) with $\xi_1 \to \theta$, $\xi_2 \to \phi$ and with

$$G^{11} = 2b, \quad G^{22} = a + 2\mathrm{ctg}^2\theta + U(\theta, \phi), \quad G^{12} = G^{21} = 0, \tag{5.9}$$

$$G = \det G^{\mu\nu} = 2b \left(a + 2\mathrm{ctg}^2\theta + U(\theta, \phi) \right). \tag{5.10}$$

Expanding (2.7) and comparing the obtained formula with (5.7), we find the components of the external field A_μ,

$$A_1(\theta, \phi) = 0 \qquad A_2(\theta, \phi) = \frac{c - 2mU(\theta, \phi)}{2 \left(a + 2\mathrm{ctg}^2\theta + U(\theta, \phi) \right)}, \tag{5.11}$$

and the potential of the model

$$V(\theta, \phi) = -\frac{(c - 2mU(\theta, \phi))^2}{4 \left(a + 2\mathrm{ctg}^2\theta + U(\theta, \phi) \right)} + m^2 U(\theta, \phi)$$

$$+ \frac{2b\mathrm{ctg}\theta \left((a - 2 + U(\theta, \phi)) \sin 2\theta + \sin^2 \theta \frac{\partial U(\theta, \phi)}{\partial \theta} \right) + \sin^2 \theta \left(\frac{\partial U(\theta, \phi)}{\partial \phi} \right)^2}{4\sqrt{a \sin^2 \theta + 2 \cos^2 \theta + \sin^2 \theta \cdot U(\theta, \phi)}}$$

$$- \frac{3 \left(\frac{\partial U(\theta, \phi)}{\partial \phi} \right)^2}{16\sqrt{a \sin^2 \theta + 2 \cos^2 \theta + \sin^2 \theta \cdot U(\theta, \phi)}}$$

$$+ \frac{1}{4\sqrt{a \sin^2 \theta + 2 \cos^2 \theta + \sin^2 \theta \cdot U(\theta, \phi)}} \frac{\partial^2 U(\theta, \phi)}{\partial \phi^2}$$

$$- \frac{6b \left((a - 2 + U(\theta, \phi)) \sin 2\theta + \sin^2 \theta \frac{\partial U(\theta, \phi)}{\partial \theta} \right)^2}{16 \left(a \sin^2 \theta + 2 \cos^2 \theta + \sin^2 \theta \cdot U(\theta, \phi) \right)^{\frac{3}{2}}}$$

$$+ \frac{2b \left(2\frac{\partial U(\theta, \phi)}{\partial \theta} \sin 2\theta + 2 (a - 2 + U(\theta, \phi)) \cos 2\theta + \sin^2 \theta \frac{\partial^2 U(\theta, \phi)}{\partial \theta^2} \right)}{\sqrt{a \sin^2 \theta + 2 \cos^2 \theta + \sin^2 \theta \cdot U(\theta, \phi)}} \tag{5.12}$$

Formulas (5.11) and (5.12) complete the construction of quasi-exactly solvable models with hermitian hamiltonians having an infinite number of exact solutions.

References

[1] O.V. Zaslavsky and V.V. Ulyanov, Sov. Phys. - JETP **60** 991 (1984)

[2] A.V. Turbiner and A.G. Ushveridze, Phys. Lett. **126A**, 181 (1987)

[3] A.G. Ushveridze, Sov. Phys. - Lebedev Inst. Rep. **2** 50 (1988)

[4] A.V. Turbiner, Comm. Math. Phys. **118**, 467 (1988)

[5] A.G. Ushveridze, Sov. J. Part. Nucl. 20, 504 (1989)

[6] M.A. Shifman, Int J. Mod. Phys. A **4**, 2897 (1989)

[7] A.Y.Morozov, A.M.Perelomov,A.A.Rosly, M.A.Shifman and A.V.Turbiner, Int. J. Mod. Phys. A **5** 803 (1990)

[8] A.G. Ushveridze, Sov. J. Part. Nucl. **23**, 25 (1992)

[9] A.González-López, N Kamran and P.J.Olver, American J. Math. **114** 1163–85,(1992)

[10] A.González-López , N Kamran and P.J.Olver Commun. Math. Phys. **153** 117–46,(1993)

[11] M.A.Shifman, "Quasi-exactly solvable spectral problems and conformal field theory", in: *Lie Algebras, Cohomologies and New Findings in Quantum Mechanics*, Contemporary Mathematics, v.160, pp. 237-262, 1994; AMS, N. Kamran and P. Olver (eds).

[12] A.G. Ushveridze, *Quasi-Exactly Solvable Problems in Quantum Mechanics*, IOP Publishing, Bristol (1994)